Math 3210 Tutorial 1

Introduction + Review of linear algebra

Linear Constrain maximization

$$
\begin{array}{ll}
\text{optimize} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \le b_1, \\ \quad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \le b_m, \end{cases} \\
\text{where} & x_i \ge 0, \quad i = 1, 2, \cdots, n.
\end{array}
$$

$x_1, \cdots, x_n$ as portion/holdings of different stocks

Consider $c_1, \cdots c_n$ as return on investment

$\left.\begin{array}{l} a_{11} \cdots a_{1n} \\ a_{m1} \cdots a_{mn} \end{array}\right\} \to$ exposure to risk factors of each stock.

General concepts:

Create new variables to tackle the corner points.

$Z = c_1x_1 + \cdots + c_nx_n + 0s_1 \cdots 0 \cdot s_m.$

$a_{11}x_1 + \cdots + a_{1n}x_n + s_1 = b_1$

$a_{21}x_1 + \cdots + a_{2n}x_n + 0 + s_2 = b_2$

$a_{m1}x_1 + \cdots + a_{mn}x_n + \cdots + s_m = b_m$

$x_1, \cdots, x_n, s_1, \cdots, s_m \ge 0.$

Example 1: Solution set to a system of linear equation

Determine if the two following system has a nontrivial solution, if no find the set of solution:

1a)

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 2x_2 + 4x_3 = 0$$
$$6x_1 + 2x_2 - 8x_3 = 0$$

pivot values

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -\frac{4}{3} & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - \frac{4}{3}x_3 = 0$ $\quad x_1 = \frac{4}{3}x_3$

$x_2 = 0$ $\quad x_2 = 0$

$x_3 = x_3$

$$X = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} x_3 = Vx_3 \quad V = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

free variables

1b)

$$10x_1 - 3x_2 - 2x_3 = 0 \Rightarrow x_1 - 0.3x_2 - 0.2x_3 = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 = 0.3x_2 + 0.2x_3$

$x_2 = x_2$

$x_3 = x_3$

$$X = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} //$$

Example 2: Describe all solution of **Ax=b**, where

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}, b = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & 8 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - \frac{4}{3}x_3 = -1 \qquad x_1 = -1 + \frac{4}{3}x_3$$
$$x_2 = 2 \qquad x_2 = 2$$
$$0 = 0. \qquad x_3 = \cancel{x_3}\ x_3$$

$$X = \underset{\downarrow \atop p}{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}} + x_3 \underset{\downarrow \atop v}{\begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}}$$

Column Space, Rank and Null Space:

Column space of matrix A is the set Col A of all linear combinations of the columns of A

The pivot columns of a matrix A form a basis for the column space of A

Rank

Number of linearly independent column

Dimension of column space

Example 3:

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$

Determine the dimension of the column space of A and whether b is in the column space of A.

$$\begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -3 & -4 \\ 0 & \boxed{-6} & -18 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} 3 \\ 15 \\ 0 \end{matrix}$$

$\operatorname{rank}(A) = 2$.

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Null Space / Kernel

The Null Space of a matrix A is the set **Nul A** of all solution to the homogeneous equation **AX=0**.

Example 4: Find a basis of the null space of the matrix:

$$A = \begin{pmatrix} -3 & 6 & -1 & 1-7 \\ 1 & -2 & 2 & 3-1 \\ 2 & -4 & 5 & 8-4 \end{pmatrix}$$

$$A \sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\cancel{x_1 - 2x_2 - x_4 + 3x_5 = 0}$

$x_1 - 2x_2 - x_4 + 3x_5 = 0 \Rightarrow x_1 = 2x_2 + x_4 - 3x_5$

$x_3 + 2x_4 - 2x_5 = 0 \quad x_3 = -2x_4 + 2x_5$

$0 = 0.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Definition:

Let A be a matrix with **N columns** with a **Dim Col = M**

Dimension of kernel of A = ~~**N-N**~~

N-M

Example 5: Find the Dimension of the kernel of the matrix B

$$B = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Basic Solution to a System:

Example 6:

In the case of Ax=B, rank (A) =m<n, if we want to find a solution by setting (n-m) variables to be 0 and solving for the remaining m variables.

$$A = \begin{pmatrix} 1 & 4 & 0 & 6 \\ 2 & 3 & 3 & 1 \end{pmatrix}, B = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 0 & 6 \\ 2 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

Set x_3, x_4 =0

See a clearer picture

$$\begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 4 \\ 3 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 3 \end{matrix}} & \boxed{\begin{matrix} 6 \\ 1 \end{matrix}} \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \end{array} \begin{pmatrix} \boxed{x_1} \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_1 + 4x_2 + 0(x_3) + 6x_4 \\ 2x_1 + 3x_2 + 3x_3 + x_4 \end{bmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 3 \end{pmatrix} x_3 + \begin{pmatrix} 6 \\ 1 \end{pmatrix} x_4 = \begin{pmatrix} 9 \\ 8 \end{pmatrix}.$$

$$x_3, x_4 = 0$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} x_2 = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

Example 7: 2D constrain maximization

Maximize:

$z = 2x_1 + 3x_2$

Subject to:

$2x + y \leq 4$

$x + 2y \leq 5$

$x, y \geq 0$

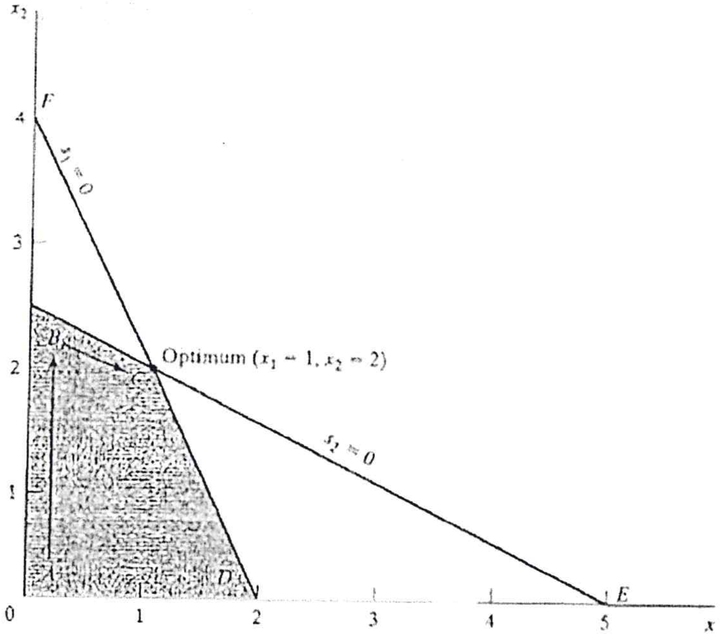
Add in new variables:

$2x + y + s_1 + 0_2 \cancel{\leq} = 4$

$x + 2y + 0s_1 + s_2 \cancel{\leq} = 5$

$x, y, s_1, s_2 \geq 0$

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$



x_2
F
4
3
2
1
0
$s_1 = 0$
B
C
Optimum ($x_1 = 1, x_2 = 2$)
$s_2 = 0$
A
D
E
1
2
3
4
5
x